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# A THEORETICAL APPROACH TO DETERMINATION OF THE THERMAL DIFFUSIVITY OF SOLIDS BY USING A PERIODIC HEATING TECHNIQUE

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The present work is part of a program aimed at development of an experiment to determine the thermal diffusivity of solids by using the method of periodic heating. The space-time heat conduction equation is solved analytically in the form of an infinite series for a slab with a uniform initial temperature. One of the slab surfaces is maintained at a constant temperature, while the other is subjected to a periodic temperature. We have proved numerically that the series can be truncated to a few terms according to the slab thickness, giving a negligible error. On the basis of the discussed theoretical approach, the experimental requirements are suggested for determination of the thermal diffusivity of solids.

The measurement of thermal diffusivity is one of the important areas associated with the experimental determination of the thermophysical properties of materials. Such measurements may be mainly classified under two techniques: dynamic and static categories.

The dynamic technique is more flexible than the static one, which requires long time periods (especially in the case of materials with low thermal conductivity) to reach thermal equilibrium. Many methods are classified under this technique [1–5].

In the present work, a novel theoretical approach is adopted for measurement of the thermal diffusivity of solids, whether conductors or insulators, by using the periodic heating method. The space-time dependent heat conduction equation is solved analytically in an infinite series form. It has been shown numerically that the series can be truncated to a few terms with negligible error.

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**Theoretical model**

In the proposed measurement, the specimen is a slab of thickness $H$ subjected to a steady-state periodic temperature at one surface and maintained at a constant temperature at the other surface. It is required to obtain the temperature distribution at any position within the slab, from which one can determine experimentally the thermal diffusivity of the slab material. The periodic temperature is considered to be a square wave of period $\tau$ and amplitude $\Delta T$, superimposed on a steady temperature level $T_{av}$. The heat conduction equation is solved under the following considerations:

(i) one–dimensional heat flow in a direction perpendicular to the slab plane surface,

(ii) constant thermal diffusivity within the temperatures used.

The space-time dependent heat conduction equation is given by

$$\frac{\partial T(x,t)}{\partial t} = \lambda \frac{\partial^2 T(x,t)}{\partial t^2} \; ; \qquad 0 \le x \le H \, , \; t > 0 \tag{1}$$

where $\lambda$ is the thermal diffusivity of the slab material, $H$ is the slab thickness, and $T$ is the temperature at any position $x$ in the slab and at any time $t$ measured from the beginning of the latest positive period.

The initial and boundary conditions are

$$T(x,0) = T_{av} \; ; \qquad 0 \le x \le H \tag{2–a}$$

$$T(0,t) = T_{av} \; ; \qquad t > 0 \tag{2–b}$$

and

$$T(H,t) = \begin{cases} T_{av} + \Delta T ; & 0 < t < \tau/2 \\ T_{av} - \Delta T ; & \tau/2 < t < \tau \end{cases} \tag{2–c}$$

**Analytical solution**

Equation (1) can be solved analytically by considering the temperature $T(x, t)$ as a sum of two space-time dependent functions, $P(x, t)$ and $Q(x, t)$, such that

$$T(x,t) = P(x,t) + Q(x,t) \tag{3}$$

on the condition that $P$ and $Q$ satisfy the following relations:

$$\frac{\partial P}{\partial t} = \lambda \frac{\partial^2 P}{\partial x^2} \tag{4}$$

with initial and boundary conditions

$$P(0,t) = 0, \qquad P(H,t) = 0; \qquad t > 0 \tag{5-a}$$

$$P(x,0) = T_{av}; \qquad 0 \le x \le H \tag{5-b}$$

and

$$\frac{\partial Q}{\partial t} = \lambda \frac{\partial^2 Q}{\partial x^2} \tag{6}$$

with initial and boundary conditions

$$Q(0,t) = T_{av}; \qquad t > 0 \tag{7-a}$$

$$Q(x,0) = 0; \qquad 0 \le x \le H \tag{7-b}$$

and

$$Q(H,t) = \begin{cases} T_{av} + \Delta T; & 0 < t < \tau/2 \\ T_{av} - \Delta T; & \tau/2 < t < \tau \end{cases} \tag{7-c}$$

Applying a Fourier series [6], Eq. (4) has been solved, giving $P(x,t)$ as

$$P(x,t) = \frac{4T_{av}}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)} \sin\left\{\frac{(2n+1)\pi x}{H}\right\} e^{-\frac{\lambda(2n+1)^2\pi^2 t}{H^2}} \tag{8}$$

Again, function $Q(x,t)$ can be assumed to be a sum of two functions, $U(x)$ and $V(x,t)$, such that

$$Q(x,t) = U(x) + V(x,t) \tag{9}$$

where $U(x)$ adn $V(x, t)$ satisfy the relations

$$\frac{d^2U}{dx^2} = 0 \tag{10}$$

with the conditions

$$U(O) = T_{av} \tag{11-a}$$

$$U(H,) = \begin{cases} T_{av} + \Delta T; & 0 < t < \tau/2 \\ T_{av} - \Delta T; & \tau/2 < t < \tau \end{cases} \tag{11–b}$$

and

$$\frac{\partial V(x,t)}{\partial t} = \lambda \frac{\partial^2 V(x,t)}{\partial x^2} \tag{12}$$

with the conditions

$$V(0, t) = 0, \quad \text{and} \quad V(H, t) = 0; \quad t > 0 \tag{13–a}$$

and

$$V(x,0) = - U(x); \qquad 0 \le x \le H \tag{13–b}$$

Equation (10) is solved directly, yielding

$$U(x) = T_{av} \pm \Delta T\left(\frac{x}{H}\right) \tag{14}$$

taking the positive sign for $0 < t < \tau/2$, and the negative sign for $\tau/2 < t < \tau$

Equation (12) is solved in the same manner as Eq. (4) and yields

$$V(x,t) = -\sum_{n=0}^{\infty} \frac{4T_{av}}{(2n+1)\pi} \sin\left\{\frac{(2n+1)\pi x}{H}\right\} e^{-\frac{\lambda(2n+1)^2\pi^2 t}{H^2}}$$

$$\pm \sum_{n=0}^{\infty} \frac{2(\Delta T)}{n\pi} (-1)^n \sin\left(\frac{n\pi x}{H}\right) e^{-\frac{\lambda n^2 \pi^2 t}{H^2}} \tag{15}$$

Now, using Eqs (8), (14) and (15) in Eqs (3) and (9), the temperature $T$ at any position $x$ in the slab and at any time $t$ can be written as

$$T(x,t) = T_{av} \pm \left\{ \Delta T \left(\frac{x}{H}\right) + \sum_{n=1}^{\infty} \frac{2(\Delta T)}{n\pi} (-1)^n \sin\left(\frac{n\pi x}{H}\right) e^{-\frac{\lambda n^2 \pi^2 t}{H^2}} \right\} \quad (16)$$

where again the positive sign is considered for $0 < t < \tau/2$, while the negative one is considered for $\tau/2 < t < \tau$.

## Results and discussion

Equation (16) can be used to give the temperature $T$ at any position $x$ and time $t$. It is clear that the series is divergent because of the exponential term, and hence it can be truncated to finite terms. This was performed numerically.

It was found that, for a specimen of thickness 1 cm and thermal diffusivity $0.32 \cdot 10^{-4}\, m^2 \cdot s^{-1}$, the summation can be truncated to one term ($n = 1$), giving a relative error $6.67 \cdot 10^{-5}$, which may be neglected. This yields that, for a specimen thickness ≤1 cm, Eq. (16) can be rewritten as

$$T(x,t) = T_{av} \pm \left\{ \Delta T \left(\frac{x}{H}\right) - \frac{2(\Delta T)}{\pi} \sin\left(\frac{\pi x}{H}\right) e^{-\frac{\lambda \pi^2 t}{H^2}} \right\} \quad (17)$$

For the mentioned specimen, the temperature $T(x, t)$ is calculated through Eq. (17) at different positions $x$ and times $t$. Figure 1 illustrates a sample of the results obtained at the center of the specimen. Exactly similar graphs are obtained at different positions in the specimen.

From Fig. 1, it is noted that after a time $t_c$ the temperature $T(x, t)$ becomes constant, i.e. the summation term vanishes. The time $t_c$ was calculated at different positions in the slab and was found to have a peak value at the slab center; its value then decreased towards the slab surfaces in a symmetrical form, as shown in Fig. 2.

The explained procedure was repeated for different thicknesses of the specimen. It was found that only a few terms may be taken from the series without exceeding a very small error. The results obtained are summarized in Table 1.

For all thicknesses, the temperature $T(x,t)$ was calculated and the resulting shape was found to be exactly as illustrated in Fig. 1. The only difference is that the time $t_c$ at which $T$ becomes constant increases with the specimen thickness, which is to be expected. The results obtained are included in Fig. 2.

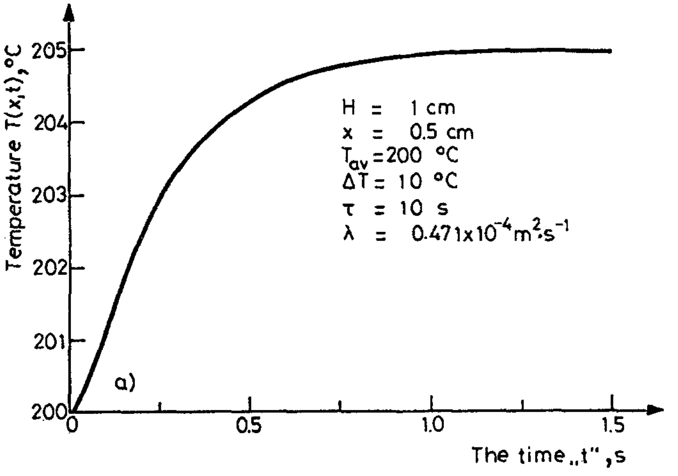


**Fig. 1–a** Variation of temperature $T(x, t)$ in the transient period

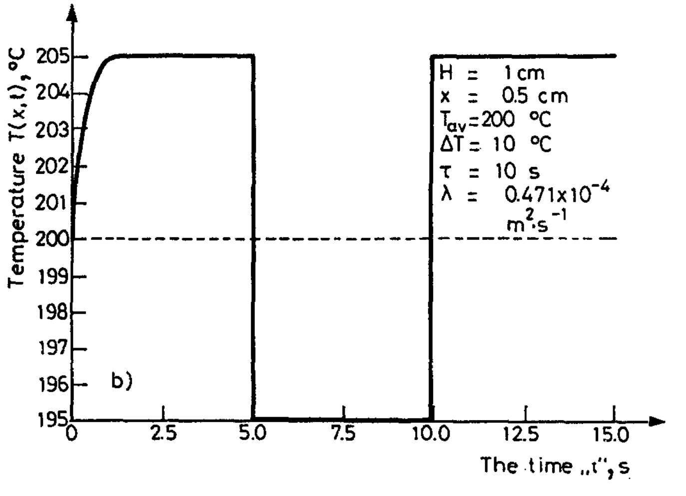


**Fig. 1–b** Variation of the temperature $T(x, t)$ with time

From Fig. 2, a peak is noted in all curves, giving a maximum time $t_{c,\ mx}$ at the center of the specimen. It is also noted that the value of $t_{c,\ mx}$ increases with the thickness.

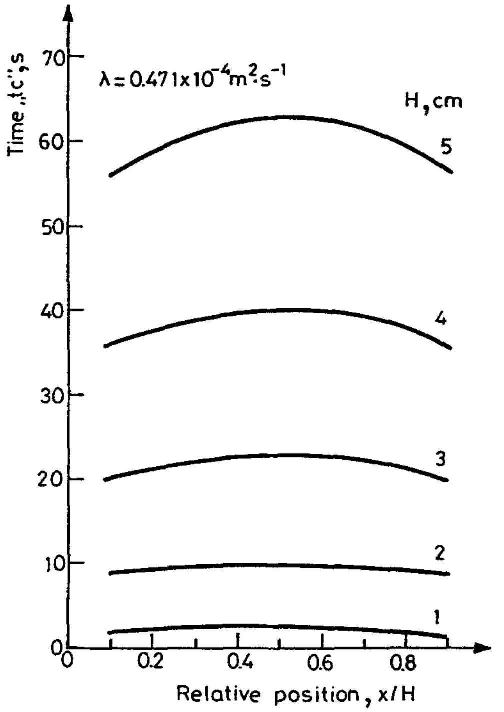


**Fig. 2** Variation of time $t_c$ with position

**Table 1** Specimen thickness with number of terms in the series

| $H$, cm | $n$ | Relative error |
|---|---|---|
| 1 | 1 | $7.67\cdot10^{-5}$ |
| 2 | 3 | $7.18\cdot10^{-6}$ |
| 3 | 4 | $2.19\cdot10^{-4}$ |
| 4 | 5 | $9.98\cdot10^{-4}$ |
| 5 | 6 | $2.31\cdot10^{-3}$ |

This is illustrated in Fig. 3.

The discussed procedure was repeated for some materials including insulators (e.g. glass: $\lambda = 0.0655\ m^2\cdot s^{-1}$) and conductors (e.g. copper: $\lambda = 0.1119\ m^2\cdot s^{-1}$), in order to study the effect of the diffusivity on $n$, $t_c$ and

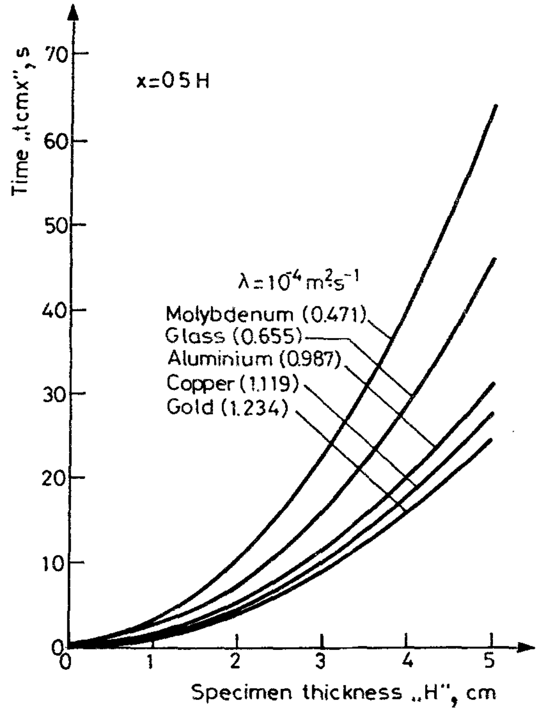


**Fig. 3** Variation of the time $t_{c,\ mx}$ with the thickness

$t_{c,\ mx}$. The results obtained showed that the time $t_c$, and consequently $t_{c,\ mx}$, increases as the diffusivity decreases. This is summarized in Fig. 3.

## Outline of the proposed experiment

Figure 4 shows a schematic layout of the proposed experiment. The disk-shaped specimen of thickness ≤1 cm is heated from the bottom surface by a heating coil fed with a square wave voltage. The upper surface of the specimen is maintained at a constant temperature $T_{av}$ by using a condenser connected to a high capacity thermostat. The resulting temperature wave at a certain specified position within the material is detected by using a precalibrated thermocouple.

The input heating wave and the resulting wave (at the specified position $x$ from the upper surface) are recorded simultaneously on a double $x$-$y$ chart recorder. Having the value of the temperature $T(x, t)$ at a certain time $t$ from the beginning of the input square wave, the thermal conductivity can be determined by using Eq. (17), which can be rewritten as

$$\lambda = -\frac{H^2}{\pi^2 t} \ln\left\{ \left[ \pm \frac{2(\Delta T)}{\pi} \sin\left(\frac{\pi x}{H}\right) \right]^{-1} \cdot \left[ T_{av} \pm \Delta T \left(\frac{x}{H}\right) - T(x,t) \right] \right\} \quad (18)$$

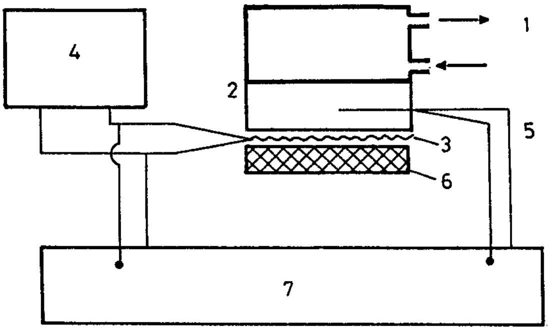


**Fig. 4** Schematic layout of the proposed experiment : 1. Inlet and outlet water from the thermostat; 2. The specimen; 3. The heating coil; 4. The source of the square wave; 5. Thermocouple; 6. Insulator; 7. Double $x$–$y$ chart recorder

from which the thermal diffusivity can be obtained directly, taking the positive sign for $0 < t < \tau/2$ and the negative sign for $\tau/2 < t < \tau$.

## Conclusion

The present theoretical approach is the basis for an experimental measurement of the thermal diffusivity through measurement of the temperature $T(x, t)$ at a specified position in the specimen. A specimen thickness ≤1 cm is preferred for attainment of the boundary conditions of the theoretical work and the direct determination of the thermal diffusivity $\lambda$ by using Eq. (18); otherwise, for thickness >1 cm, $\lambda$ can be obtained from Eq. (16) through a computer program after taking the number of terms of the series according to the thickness with the aid of Table I. In all cases, the temperature $T(x,t)$ must be measured at a time $t < t_c$, because after this time the temperature becomes constant and the time does not affect its value.

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**Zusammenfassung** — Vorliegende Arbeit ist Teil eines Experimenteprogrammes zur Bestimmung der Temperaturleitfähigkeit von Feststoffen mittels der Methode des periodischen Erhitzens. Die Raumzeit-Gleichung für die Wärmeleitfähigkeit wurde für eine Platte mit homogener Ausgangstemperatur analytisch in Form einer unendlichen Reihe gelöst. Eine der Plattenoberflächen wird auf konstanter Temperatur gehalten, während die andere einer periodischen Temperatur ausgesetzt wird. Es konnte gezeigt werden, daß die Reihen je nach Dicke der Platte nach einigen Gliedern abgebrochen werden können, ohne dabei einen unvernachlässigbaren Fehler zu begehen. Auf der Grundlage der theoretischen Ergebnisse wurden experimentelle Anforderungen zur Bestimmung der Temperaturleitfähigkeit von Feststoffen erarbeitet.